REFERENCES

1. Ufliand, Ia. S., Integral Transforms in Elasticity Theory Problems. "Nauka", Leningrad, 1967.
2. Aleksandrov, V. M., On the approximate solution of some integral equations of the theory of elasticity and mathematical physics. PMM Vol. 31, №6, 1967.
3. Aleksandrov, V. M., Asymptotic methods in contact problems of elasticity theory. PMM Vol. 32, №4, 1968.
4. Aleksandrov, V. M., Babeshko, V. A. and Kucherov, V. A., Contact problems for an elastic layer of slight thickness. PMM Vol. 30, №1, 1966.
5. Sneddon, I., Fourier Transforms. Moscow, Izd. Inostr. Lit., 1955.
6. Gradshtein, I. S. and Ryzhik, I. I., Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moscow, 1967.

Translated by M. D. F.

UDC 539.3

ON THE LOCAL AXISYMMETRIC COMPRESSION OF AN ELASTIC LAYER WEAKENED BY AN ANNULAR OR CIRCULAR CRACK

PMM Vol. 38, №1, 1974, pp. 139-144

V. S. NIKISHIN and G. S. SHAPIRO
(Moscow)

(Received May 8, 1973)

Two kindred problems on the compression of an elastic layer by a local load applied symmetrically to its surfaces are considered.

In one case the layer has an annular crack with inner radius a and outer radius b on the middle plane. The quantities a and b $(0 < a < b)$ are selected from the condition that the annular crack subjected to a load would be opened up and a normal tensile stress concentration would originate on the circumferential contours $r = a$ and $r = b$.

In the other case, the layer has a circular crack of radius b on the middle plane. Under the effect of a load in a circular domain of radius a $(a < b)$ the crack edges will be in contact, and will separate from each other in the annular region $a < r < b$. The quantity a is unknown and to be determined from the condition that the contact pressure on the circumferential contour $r = a$ is zero; the quantity b is selected from the condition that a normal tensile stress concentration would originate on the contour $r = b$.

In both cases the crack lips are assumed smooth. The crack is a mathematical slit in the unloaded layer.

In the general case, the layer is compressed under the effect of an arbitrary local load applied to its upper and lower boundary planes symmetrically relative to the axis and the middle plane. As an illustration, the particular case of compression of the layer by two normal concentrated forces directed along the axis of symmetry of the problem is considered (Fig. 1).

The problems of annular and circular cracks in an infinite layer were considered in another formulation in [1 — 5]. It was assumed therein that the layer is stretched under the effect of a load applied directly to the crack edges.

1. Construction of the general solution of the problem of elasticity theory for a layer. The axisymmetric problem under consideration is solved in the dimensionless variables $\rho = r / b$ and $t = z / H$ ($2H$ is the layer thickness). The origin of the ρ, t coordinate system is taken at the center of the annular or circular crack on the middle plane.

The general solution of the elasticity theory problem is constructed by using the Love stress functions and is represented in terms of the Hankel integral [6]. The normal displacement is represented by the formula

$$\frac{E}{(1+\nu) b} w(\rho, t) = \int_0^\infty \Delta_w(t, \beta) J_0(\rho\beta)\, d\beta \qquad (1.1)$$

where $\Delta_w(t, \beta)$ is expressed in terms of four arbitrary functions $A(\beta)$, $B(\beta)$, $C(\beta)$, $D(\beta)$. The stresses σ_r, σ_θ, σ_z, τ_{rz} and the radial displacement u are represented analogously.

Let the tangential stresses be zero on the upper and lower boundary planes of the layer $t = \pm 1$:

$$\tau_{rz}(\rho, t)_{t=\pm 1} = 0, \quad 0 \leqslant \rho < \infty \qquad (1.2)$$

and the normal stresses are represented as

$$\sigma_z(\rho, t)|_{t=\pm 1} = p_1(\rho) + p_1^*(\rho), \qquad 0 \leqslant \rho < \infty \qquad (1.3)$$

Here $p_1(\rho)$ is an arbitrarily specified function of the normal load intensity, and $p_1^*(\rho)$ is some function of the intensity of a small additional load, the need for whose introduction will be clarified below. For the symmetric external load (1.2), (1.3) the shear stresses in the middle plane of the layer with an annular or circular crack with smooth edges are zero

$$\tau_{rz}(\rho, t)|_{t=0} = 0, \qquad 0 \leqslant \rho < \infty \qquad (1.4)$$

and the normal stresses are represented in terms of some still unknown function $p_0(\rho)$

$$\sigma_z(\rho, t)|_{t=0} = p_0(\rho), \qquad 0 \leqslant \rho < \infty \qquad (1.5)$$

Let us represent the functions $p_i(\rho)$ $(i = 0, 1)$ in terms of the Hankel integral

$$p_i(\rho) = \int_0^\infty \beta \bar{p}_i(\beta) J_0(\rho\beta)\, d\beta \qquad (1.6)$$

$$\bar{p}_i(\beta) = \int_0^\infty \rho p_i(\rho) J_0(\rho\beta)\, d\rho \qquad (i = 0, 1) \qquad (1.7)$$

The function $p_1^*(\rho)$ is represented by the Hankel integral of the special form

$$p_1^*(\rho) = \int_0^\infty \beta f(\beta) \Delta_w(1, \beta) J_0(\rho\beta)\, d\beta \qquad (1.8)$$

where $\Delta_w(1, \beta)$ is the integrand in the representation of the normal displacements (1.1) for $t = 1$ and $f(\beta)$ is some arbitrary continuous function which will be indicated below.

The unknown functions $A(\beta)$, $B(\beta)$, $C(\beta)$, $D(\beta)$, in the general solution (1.1) are expressed in terms of $f(\beta)$ and the transform $\bar{p}_i(\beta)$ $(i = 0, 1)$ from conditions (1.2)–(1.5) for the upper half of the layer taking account of (1.6)–(1.8).

Let us write expressions for the functions $\Delta_w(t, \beta)$ for $t = 1$ and $t = 0$

$$\Delta_w(1, \beta) = 2(1-\nu)[\Delta_{11}(\beta)\bar{p}_1(\beta) - \Delta_{10}(\beta)\bar{p}_0(\beta)] \qquad (1.9)$$
$$\Delta_w(0, \beta) = 2(1-\nu)[\Delta_{01}(\beta)\bar{p}_1(\beta) - \Delta_{00}(\beta)\bar{p}_0(\beta)]$$
$$\Delta(\beta)\Delta_{11}(\beta) = 1 - e^{-4\beta\lambda} + 4\beta\lambda e^{-2\beta\lambda}$$
$$\Delta(\beta)\Delta_{10}(\beta) = \Delta(\beta)\Delta_{01}(\beta) = 2e^{-\beta\lambda}[1 - e^{-2\beta\lambda} + \beta\lambda(1 + e^{-2\beta\lambda})]$$
$$\Delta(\beta)\Delta_{00}(\beta) = \Delta(\beta)\Delta_{11}(\beta) - 2(1-\nu)f(\beta)(1 - e^{-2\beta\lambda})^2$$
$$\Delta(\beta) = (1 - e^{-2\beta\lambda})^2 - 4(\beta\lambda)^2 e^{-2\beta\lambda} - 2(1-\nu)f(\beta)(1 - e^{-4\beta\lambda} + 4\beta\lambda e^{-2\beta\lambda})$$

Furthermore, the following formulas for the normal stresses and displacements on the middle plane of the layer are required:

$$\sigma_z(\rho, t)|_{t=0} = \int_0^\infty \beta\bar{p}_0(\beta) J_0(\rho\beta)\, d\beta \qquad (1.10)$$

$$\frac{E}{2(1-\nu^2)b} w(\rho, t)|_{t=0} = \int_0^\infty \Delta_{01}(\beta)\bar{p}_1(\beta) J_0(\rho\beta)\, d\beta - \int_0^\infty \Delta_{00}(\beta)\bar{p}_0(\beta) J_0(\rho\beta)\, d\beta \qquad (1.11)$$

The convergence of each integral in (1.11) is required separately for arbitrary $p_i(\rho)$ $(i = 0, 1)$ representable by Hankel integrals. For $f(\beta) = 0$ and therefore, for $p_1^*(\rho) = 0$, both integrals of (1.11) diverge at the lower limit since we have $\Delta_{ij}(\beta) \sim 6(\beta\lambda)^{-3}$ $(i, j = 0, 1)$ and $\bar{p}_i(0) = F/2\pi$ $(i = 0, 1)$, as $\beta \to 0$, where F is the resultant force of the normal stresses on the $t = 0$ and $t = 1$ planes. The convergence of both integrals in (1.11) at the lower limit will be assured by the function $f(\beta)$ introduced uniquely for this purpose in terms of the additional load $p_1^*(\rho)$ (1.8). For the integrals in (1.11) to converge it is sufficient to require that $f(0) \neq 0$. Moreover, it is required of $f(\beta)$ that the function $\Delta(\beta)$ (the last relationship in (1.9)) should not vanish on the whole half-axis $0 \leqslant \beta < \infty$ and that the intensity of the additional load $p_1^*(\rho)$ for every $\rho \in [0, \infty)$ and its resultant force F^* should be sufficiently small in absolute value. The function

$$f(\beta) = -\varepsilon(k^2 + \beta^2)^{-1/2} e^{-n\beta} \qquad (1.12)$$

(ε, k, n are positive constants), for example, satisfies all these requirements.

All integrands of the general solution of the form (1.1) expressed in terms of $f(\beta)$ and $\bar{p}_i(\beta)$ $(i = 0, 1)$ by the method described above, are continuous and bounded on the whole half-axis $0 \leqslant \beta < \infty$. Let the modulus $\Delta_w(1, \beta)$ have the upper bound $M > 0$, then we find the estimate

$$|p_1^*(\rho)| < \varepsilon n^{-2} k^{-3} M \quad (0 \leqslant \rho < \infty), \qquad |F^*| < 2\pi\varepsilon k^{-3} M$$

from the integral representation (1.8) and its inverse by taking account of (1.12). For sufficiently small ε and large k, n the modulus of $p_1^*(\rho)$ for all $\rho \in [0, \infty]$ and the modulus of F^* can be made negligibly small.

2. Problems on annular and circular cracks. On the middle plane of the layer let there be an open annular crack $\rho^* < \rho < 1$ ($\rho^* = a / b$) or a circular crack $0 \leqslant \rho < 1$ open in the annulus $\rho^* < \rho < 1$. In both cases the boundary conditions on the middle plane $t = 0$ of the layer are written as

$$\tau_{rz}(\rho, t)|_{t=0} = 0, \qquad 0 \leqslant \rho < \infty \tag{2.1}$$

$$w(\rho, t)|_{t=0} = 0, \qquad 0 \leqslant \rho < \rho^*, \quad 1 < \rho < \infty \tag{2.2}$$

$$\sigma_z(\rho, t)|_{t=0} = 0, \qquad \rho^* < \rho < 1 \tag{2.3}$$

The boundary condition (2.1) was taken into account in constructing the general solution of the problem in Section 1. Substituting (1.10) and (1.11) into the boundary conditions (2.2) and (2.3), we obtain dual integral equations for the unknown transform $\bar{p}_0(\beta)$

$$\int_0^\infty \Delta_{00}(\beta)\bar{p}_0(\beta) J_0(\rho\beta)\, d\beta = \int_0^\infty \Delta_{01}(\beta)\bar{p}_1(\beta) J_0(\rho\beta)\, d\beta \tag{2.4}$$

$$0 \leqslant \rho < \rho^*, \quad \rho > 1$$

$$\int_0^\infty \beta\bar{p}_0(\beta) J_0(\rho\beta)\, d\beta = 0, \qquad \rho^* < \rho < 1$$

Let us represent $\Delta_{00}(\beta)$ as $\Delta_{00}(\beta) = 1 + \Delta_0(\beta)$ and let us give the main terms of the asymptotic formulas for the functions $\Delta_0(\beta)$ and $\Delta_{01}(\beta)$:

for $\beta \to 0$

$$\Delta_0(\beta) \backsim -1 + k^3[2(1-\nu)\varepsilon]^{-1}, \qquad \Delta_{01}(\beta) \backsim k^3[2(1-\nu)\varepsilon]^{-1}$$

for $\beta \to \infty$ $\qquad \Delta_0(\beta) \backsim 4(\beta\lambda)^3 e^{-2\beta\lambda}, \quad \Delta_{01}(\beta) \backsim 2\beta\lambda e^{-\beta\lambda}.$

The dual integral equations (2.4) are reduced to a Fredholm equation of the second kind for the new unknown function $\varphi(x)$ by the Noble [7] and Cooke [8] method

$$\varphi(x) + \frac{2}{\pi}\int_0^{\rho^*} K_1(x, t)\varphi(t)\, dt +$$

$$\frac{2}{\pi}\int_1^\infty K_2(x, t)\varphi(t)\, dt = \frac{2}{\pi}\Phi(x), \qquad 0 \leqslant x \leqslant \rho^*, \quad x \geqslant 1$$

$$K_1(x, t) = \begin{cases} \displaystyle\int_0^\infty \Delta_0(\beta)\cos(x\beta)\cos(t\beta)\, d\beta, & 0 \leqslant x \leqslant \rho^* \\[2ex] \displaystyle\frac{x}{x^2 - t^2} + \int_0^\infty \Delta_0(\beta)\sin(x\beta)\cos(t\beta)\, d\beta, & x \geqslant 1 \end{cases}$$

$$K_2(x, t) = \begin{cases} \displaystyle\frac{t}{t^2 - x^2} + \int_0^\infty \Delta_0(\beta)\cos(x\beta)\sin(t\beta)\, d\beta, & 0 \leqslant x \leqslant \rho^* \\[2ex] \displaystyle\int_0^\infty \Delta_0(\beta)\sin(x\beta)\sin(t\beta)\, d\beta, & x \geqslant 1 \end{cases}$$

$$\Phi(x) = \begin{cases} \int\limits_0^\infty \Delta_{01}(\beta)\, \bar{p}_1(\beta) \cos(x\beta)\, d\beta, & 0 \leqslant x \leqslant \rho^* \\ \int\limits_0^\infty \Delta_{01}(\beta)\, \bar{p}_1(\beta) \sin(x\beta)\, d\beta, & x \geqslant 1 \end{cases}$$

The main unknown function of the normal stress intensity on the middle plane of the layer $p_0(\rho)$ and its Hankel transform $\bar{p}_0(\beta)$ are expressed in terms of $\varphi(x)$ by means of the formulas

$$p_0(\rho, \lambda) = \begin{cases} \dfrac{\varphi(\rho^*, \lambda)}{\sqrt{\rho^{*2} - \rho^2}} - \int\limits_\rho^{\rho^*} \dfrac{\partial}{\partial x} \varphi(x, \lambda) \dfrac{dx}{\sqrt{x^2 - \rho^2}}, & 0 \leqslant \rho < \rho^* \\ 0, \quad \rho^* < \rho < 1 \\ \dfrac{\varphi(1, \lambda)}{\sqrt{\rho^2 - 1}} + \int\limits_1^\rho \dfrac{\partial}{\partial x} \varphi(x, \lambda) \dfrac{dx}{\sqrt{\rho^2 - x^2}}, & \rho > 1 \end{cases} \tag{2.5}$$

$$\bar{p}_0(\beta, \lambda) = \int\limits_0^{\rho^*} \varphi(x, \lambda) \cos(x\beta)\, dx + \int\limits_1^\infty \varphi(x, \lambda) \sin(x\beta)\, dx$$

where the dependence of these functions on the parameter $\lambda = H / b$ is stressed. All the integrals in (2.5) are continuous functions. It follows from the first formula of (2.5) that for fixed $\rho^* \in (0, 1)$ only those values of λ for which the inequalities

$$\varphi(\rho^*, \lambda) > 0, \quad \varphi(1, \lambda) > 0 \tag{2.6}$$

are satisfied, will correspond to the problem of an annular crack.

The single value $\lambda = \lambda^*$ satisfying the conditions

$$\varphi(\rho^*, \lambda^*) = 0, \quad \varphi(1, \lambda^*) > 0 \tag{2.7}$$

will correspond to the problem of a circular crack for fixed $\rho^* \in (0, 1)$.

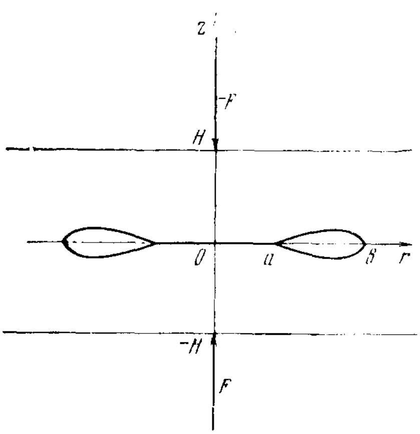

Fig. 1

Let us present the results of a numerical solution of the problem of circular and annular cracks for a layer compressed by two concentrated forces $-F$ and F $(F > 0)$ (Fig. 1). In this case $\bar{p}_1(\beta) = -F / 2\pi$. Setting $\bar{p}_1(\beta) = -1$, we obtain the result to the accuracy of the factor $F / 2\pi$. Let us assume $\rho^* = 0.5$. Graphs of the functions $\varphi^\circ = (2\pi/F)\, \varphi(\rho^*, \lambda)$ and $\varphi^\circ = (2\pi/F)\, \varphi(1, \lambda)$ (curves *1* and *2*, respectively) are presented in Fig. 2. They show that conditions (2.6) and (2.7) are satisfied for $\lambda \leqslant 0.37$. Therefore, formulation of the problems under consideration for $\rho^* = 0.5$ and $\lambda \leqslant 0.37$ is legitimate. The root $\lambda^* = 0.37$ of the function $\varphi(\rho^*, \lambda)$ corresponds to the problem of a circular crack with radius $\rho^* = 0.5$ for the area of contact of its edges. Presented in Figs. 3 and 4 for this problem are graphs of the normal stresses $\sigma_z^\circ = (2\pi/F)\sigma_z$ and the displacements $w^\circ = 2\pi E\, [(1 + \nu) b F]^{-1}\, w$ on the outer boundary of the plane and in the middle plane. The absolute value of the additional load intensity $p_1^*(\rho)$ does not exceed 0.0007.

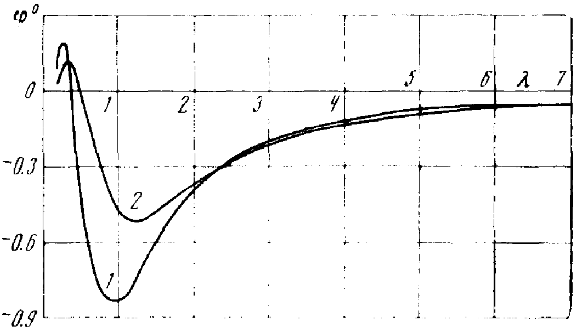

Fig. 2

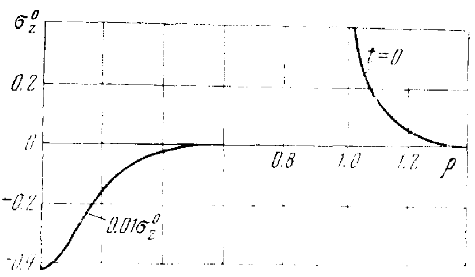

Fig. 3

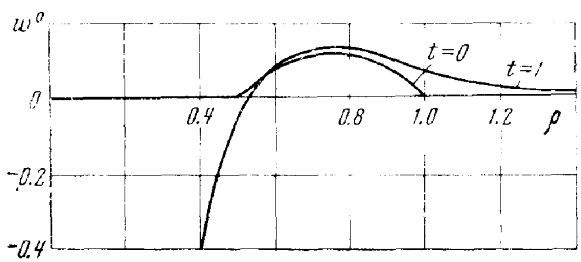

Fig. 4

REFERENCES

1. Gubenko, V. S., Problems of a circular stamp contacting a half-space, and of a layer weakened by an annular crack. Izv. Akad. Nauk SSSR, OTN, Mekh. i Mashinostr., № 55, 1961.
2. Markuzon, I. A., Equilibrium crack in a strip of finite width. Prikl. Mekh. i Tekh. Fiz., № 5, 1963.
3. Aleksandrov, V. M. and Smetanin, B. I., Equilibrium crack in a thin layer. PMM Vol. 29, № 4, 1965.
4. Pal'tsun, N. V., Stresses in an elastic layer weakened by two circular cracks. Prikl. Mekh., Vol. 32, № 2, 1967.

5. Kuz'min, Iu. N., Axisymmetric strain of an elastic layer containing coaxial circular cracks. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, №6, 1972.
6. Nikishin, V. S. and Shapiro, G. S., Three-dimensional Problems of Elasticity Theory for Multilayered Media. Computation Center Akad. Nauk SSSR, Moscow, 1970.
7. Noble, B., Certain dual integral equations. J. Math. Phys., Vol. 37, №2, 1958.
8. Cooke, J. C., Triple integral equations. Quart. J. Mech. and Applied Math., Vol. 16, №2, 1963.

Translated by M. D. F.

UDC 539.4

THE EFFECT OF A STRINGER ON THE STRESS DISTRIBUTION AROUND A HOLE

PMM Vol. 38, №1, 1974, pp. 145-153
G. T. ZHORZHOLIANI and A. I. KALANDIIA
(Tbilisi)
(Received March 23, 1973)

We investigate the effect of symmetric stringers, which reinforce a plate in the zone of a circular hole, upon the distribution of the stress field around the hole. The problem reduces to a singular integral equation of the first kind which admits an approximate examination.

An extensive literature is devoted to the problem of the transmission of forces to an elastic body through a stringer. A survey of the results obtained till 1968 is given in [1], where one can find the corresponding bibliographic data. The papers [2−7] belong to the recent investigations devoted to the theoretical aspect of the problem.

We mention that, obviously, the authors of [7] were not aware of the papers [5, 6].

1. Formulation of the problem and notation. An elastic body has the form of an infinite plate with a circular hole. Two identical elastic bars of constant cross section, situated on the same line and with ends on the circumference of the hole are attached (welded) to the plate in the radial direction. The hole is assumed to be free of applied forces. To the ends of the bars at the hole there are applied equal and opposite axial forces and the plate is subjected at infinity to uniaxial extension in the direction of the bars. We assume that the elastic medium is deformed under the conditions of generalized plane state of stress and that the reinforcing bars, called stringers from now on, are idealized one-dimensional continua, deprived of flexural rigidity. There arises the problem of the determination of the effect of the stringers on the distribution of the stresses in the plate around the hole.

For the sake of simplicity, the radius of the hole is taken to be equal to unity. We take the surface of the plate in the plane of the variable $z = x + iy$, the center of the hole in the origin and we place the axes of the stringers along the segments $[-a, -1]$ and $[1, a]$ of the real axis (Fig. 1). The algebraic value of the axial load, applied to the end of the left bar, is denoted by p_0 and the tensile force at infinity by P.

For the elements of the elastic fields and for the characteristics of the plate and the